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ON THE RATIO OF RELATIVE CONGRUENCE ZETA FUNCTIONS OF CYCLOTOMIC FUNCTION FIELDS

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Abstract. In this paper we give a determinant formula for the ratio of relative congruence zeta functions of cyclotomic function fields.

1. Introduction

Let $\mathrm{k} = \mathbb{F}_q(T)$ be the rational function field over the finite field $\mathbb{F}_q$ and $\mathbb{A} = \mathbb{F}_q[T]$. Write $\mathbb{A}^+ = \{1 \neq M \in \mathbb{A} : M \text{ is monic}\}$ and $\mathbb{A}^+_{\text{irr}} = \{P \in \mathbb{A}^+ : P \text{ is irreducible}\}$ for simplicity. For any $M \in \mathbb{A}^+$, we denote by K_M for the Mth cyclotomic function field and K_M^+ for the maximal real subfield of K_M.

It is known that there exists a polynomial $P_{K_M}(X) \in \mathbb{Z}[X]$ such that

$$\zeta(s, K_M) = \frac{P_{K_M}(q^{-s})}{(1-q^{-s})(1-q^{1-s})},$$

where $\zeta(s, K_M)$ is the congruence zeta function of K_M and $P_{K_M}(1)$ is equal to the divisor class number h_{K_M} of K_M. Let $\zeta^{(-)}(s, K_M) = \zeta(s, K_M)/\zeta(s, K_M^+)$, called the relative congruence zeta function of K_M. Then we have $\zeta^{(-)}(s, K_M) = P^{(-)}_{K_M}(q^{-s})$, where $P^{(-)}_{K_M}(X) = P_{K_M}(X)/P_{K_M^+}(X)$.

In 2010, Shiomi has expressed the polynomial $P^{(-)}_{K_M}(X)$ as the determinant of matrix up to some polynomial [5]. Recently, author and Ka gave another determinant fomula for $P^{(-)}_{K_M}(X)$ [1]. In 2007, author and Jung give determinant formulas for the ratio of class numbers of cyclotomic function fields [3]. The aim of this paper is to give an elementary determinant formula for ratio $P^{(-)}_{K_M}(X)/P^{(-)}_{K_N}(X)$ with $M, N \in \mathbb{A}^+, N|M$.

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2. Preliminaries

Let F be a finite extension of k which is contained in some cyclotomic extension K_M. Let $N \in \mathbb{A}^+$ be the conductor of F, that is, K_N is the smallest cyclotomic function field containing F. Let $\zeta(s,F)$ be the congruence zeta function of F given by

$$\zeta(s,F) = \prod_{\mathfrak{p}} \Big(1 - \frac{1}{N\mathfrak{p}^s}\Big)^{-1},$$

where $\mathfrak{p}$ runs over all primes of F. It is well known that there exists a polynomial $P_F(X) \in \mathbb{Z}[X]$ of degree $2g$, where g is the genus of F, such that

$$\zeta(s,F) = \frac{P_F(q^{-s})}{(1-q^{-s})(1-q^{1-s})}.$$

Moreover, the polynomial $P_F(X)$ satisfies $P_F(0) = 1$ and $P_F(1) = h_F$, where h_F is the divisor class number of F.

Let X_F be the group of primitive Dirichlet characters of $\mathbb{A}$ associated to F. For $\chi \in X_F$, let $L(s,\chi)$ be the L-function associated to χ given by

$$L(s,\chi) = \prod_{P \in \mathbb{A}^+_{\mathrm{irr}}} \big(1 - \chi(P) q^{-s \deg P}\big)^{-1}.$$

For any $\chi \in X_F$, let $F_\chi \in \mathbb{A}^+$ be the conductor of χ and $\tilde{\chi} = \chi \circ \pi_\chi$, where $\pi_\chi : (\mathbb{A}/N\mathbb{A})^* \to (\mathbb{A}/F_\chi\mathbb{A})^*$ is the canonical homomorphism. Let $\zeta^{(-)}(s,F) = \zeta(s,F)/\zeta(s,F^+)$ be the relative congruence zeta function of F and $P_F^{(-)}(X) = P_F(X)/P_{F^+}(X)$. Then

$$\prod_{\chi \in X_F^-} L(s,\tilde{\chi}) = P_F^{(-)}(q^{-s}) J_F^{(-)}(q^{-s}), \tag{2.1}$$

where $X_F^- = X_F \backslash X_{F^+}$ and

$$J_F^{(-)}(X) = \prod_{\chi \in X_F^-} \prod_{Q \in \mathbb{A}^+_{\mathrm{irr}},\, Q|N} \big(1 - \chi(Q) X^{\deg Q}\big).$$

To find a determinant formula for the ratio of relative congruence zeta functions, we need a variation of group determinant formula. Let G be a finite abelian group and let $L^2(G)$ be the vector space of complex-valued functions on G. Let $\widehat{G}$ be the character group of G with values in $\mathbb{C}$. For any subgroup H of G, we define $\widehat{G}^H = \{\chi \in \widehat{G} : \chi(\sigma) = 1 \text{ for all } \sigma \in H\}$ and let $\mathcal{R}_{G/H}$ be any system of representatives of G/H. When $\mathcal{R}_{G/H}$ is fixed, we define a function $r_H : G \to G$ such that $r_H(\sigma)H = \sigma H$ with $r_H(\sigma) \in \mathcal{R}_{G/H}$ for each $\sigma \in G$.

LEMMA 2.1. [3, Theorem 2.4] *For any two subgroups H, H' of G, suppose that $\mathcal{R}_{G/H}, \mathcal{R}_{G/H'}$ and $\mathcal{R}_{G/HH'}$ satisfy the condition that $r_H \circ r_{H'} = r_{H'} \circ r_H = r_{HH'}$ as functions from G to G. Then, for any $f \in L^2(G)$, we have*

$$\prod_{\chi \notin \widehat{G}^H \cup \widehat{G}^{H'}} \sum_{\sigma \in G} \chi(\sigma) f(\sigma)$$
$$= \det \left(f(\sigma\tau^{-1}) - f(\sigma r_H(\tau)^{-1}) - f(\sigma r_{H'}(\tau)^{-1}) + f(\sigma r_{HH'}(\tau)^{-1}) \right)_{\sigma,\tau},$$

where σ, τ run through $G \backslash (\mathcal{R}_{G/H} \cup \mathcal{R}_{G/H'})$.

It is easy to see that Lemma 2.1 also holds when f is a function from G to $\mathbb{Z}[X]$.

3. Ratio of the relative congruence zeta functions

From now on, we fix $M, N \in \mathbb{A}^+$ with $N|M$. Let $G_M = \mathrm{Gal}(K_M/\mathrm{k})$, $G_N = \mathrm{Gal}(K_N/\mathrm{k})$, $J = \mathrm{Gal}(K_M/K_M^+)$ and $H = \mathrm{Gal}(K_M/K_N)$. It is well-known that $G_M \cong (\mathbb{A}/M\mathbb{A})^*, G_N \cong (\mathbb{A}/N\mathbb{A})^*$ and $J \cong \mathbb{F}_q^*$. For the cyclotomic theory of function fields, we refer to [4, Chapter 12]. Under the above isomorphisms, we may identify X_{K_M} ($X_{K_M^+}$ and X_{K_N} resp.) with $\widehat{G}_M$ ($\widehat{G}_M^J$ and $\widehat{G}_M^H$ resp.) Let $\mathbb{M}_M = \{A \in \mathbb{A} : A \neq 0, \deg A < \deg M, \gcd(A, M) = 1\}$. For $\alpha \in (\mathbb{A}/M\mathbb{A})^*$, let A_α be an element of $\mathbb{M}_M$, which corresponds to α (that is, $A_\alpha + M\mathbb{A} = \alpha$). And let $\mathrm{sgn}_M(\alpha)$ be the leading coefficient of A_α and $\deg_M(\alpha) = \deg A_\alpha$. Now, we define a function $f : (\mathbb{A}/M\mathbb{A})^* \to \mathbb{Z}[X]$ by

$$f(\alpha) := \begin{cases} X^{\deg_M(\alpha)}, & \text{if } \mathrm{sgn}_M(\alpha) = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Under the isomorphism $G_M \cong (\mathbb{A}/M\mathbb{A})^*$, we also view f as a function from G_M to $\mathbb{Z}[X]$. Let $\mathbb{M}_{M,N}^* = \{A \in \mathbb{M}_M : (A)_N \in \mathbb{M}_N^+\}$, where $(A)_N$ is the element of $\mathbb{M}_N$ which satisfy $A \equiv (A)_N \pmod{N}$. Note that $\mathbb{M}_{M,N}^*, \mathbb{M}_N$ and $\mathbb{M}_N^+$ become systems of representatives of $G_M/J, G_M/H$ and G_M/JH repectively, under the map $\mathbb{M}_M \leftrightarrow (\mathbb{A}/M\mathbb{A})^* \cong G_M$. Finally, we define the matrix

$$E_{M,N}^{(-)}(X) = (X^{\deg_M(AB^{-1})} - X^{\deg_M(A(B/\mathrm{sgn}_N(B))^{-1})}$$
$$-X^{\deg_M(A(B)_N^{-1})} + X^{\deg_M(A((B)_N/\mathrm{sgn}_N(B))^{-1})})_{A,B},$$

where A, B run over $\mathbb{M}_M \backslash (\mathbb{M}^*_{M,N} \cup \mathbb{M}_N)$ and B^{-1} denote the unique element of $\mathbb{M}_M$ which satisfy $BB^{-1} \equiv 1 \pmod{M}$.

THEOREM 3.1. *For any $M, N \in \mathbb{A}^+$ with $N|M$, we have*

$$\det E^{(-)}_{M,N}(X) = \frac{P^{(-)}_{K_M}(X)}{P^{(-)}_{K_N}(X)} \frac{J^{(-)}_{K_M}(X)}{J^{(-)}_{K_N}(X)}.$$

Proof. For $\chi \in X^-_{K_M}$, as in the proof of [5, Theorem 3.1] or [2, Lemma 3], we have

$$\begin{aligned} L(s, \tilde{\chi}) &= \sum_{\alpha \in (\mathbb{A}/M\mathbb{A})^*, \operatorname{sgn}_M(\alpha)=1} \tilde{\chi}(\alpha) q^{-\deg_M(\alpha)s} \\ &= \sum_{\alpha \in (\mathbb{A}/M\mathbb{A})^*} \tilde{\chi}(\alpha) f(\alpha)|_{X=q^{-s}} \end{aligned}$$

From Lemma 2.1, we have

$$\begin{aligned} &\prod_{\chi \in X_{K_M} \backslash (X^+_{K_M} \cup X_{K_N})} L(s, \tilde{\chi}) \\ =& \left(\prod_{\chi \in X_{K_M} \backslash (X^+_{K_M} \cup X_{K_N})} \sum_{\alpha \in (\mathbb{A}/M\mathbb{A})^*} \tilde{\chi}(\alpha) f(\alpha) \right)_{X=q^{-s}} \\ =& \det \big(f(\sigma\tau^{-1}) - f(\sigma r_J(\tau)^{-1}) - f(\sigma r_H(\tau)^{-1}) + f(\sigma r_{JH}(\tau)^{-1}) \big)_{\sigma,\tau} |_{X=q^{-s}}, \end{aligned}$$

where σ, τ run through $G_M \backslash (\mathcal{R}_{G_M/J} \cup \mathcal{R}_{G_M/H})$ and we view f as a function on G_M. Thus, from (2.1), we have

$$\begin{aligned} &\det \big(f(\sigma\tau^{-1}) - f(\sigma r_J(\tau)^{-1}) - f(\sigma r_H(\tau)^{-1}) + f(\sigma r_{JH}(\tau)^{-1}) \big)_{\sigma,\tau} \\ =& \frac{P^{(-)}_{K_M}(X)}{P^{(-)}_{K_N}(X)} \frac{J^{(-)}_{K_M}(X)}{J^{(-)}_{K_N}(X)}. \end{aligned}$$

Considering the map $\mathbb{M}_M \leftrightarrow (\mathbb{A}/M\mathbb{A})^* \cong G_M$ and the definition of f, we get the result. □

REMARK 3.2. The polynomial $J^{(-)}_M(X)$ can be easily computed and $J^{(-)}_M(X) = 1$ when M is a power of an irreducible polynomial [5, Proposition 3.1].

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